A Convex Solution to Spatially-Regularized Correspondence Problems

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Spatially-Regularized Correspondence Problem

Input: Images $\mathcal{I}_1, \mathcal{I}_2 : \Omega \rightarrow \mathcal{F}$, where $\Omega \subset \mathbb{R}^2$.

Task: Find optimal $f : \Omega \rightarrow \Omega$ minimizing

$$\min_{f:\Omega\rightarrow\Omega} \int_\Omega c(x, f(x)) \sqrt{\det \begin{pmatrix} df \\ \text{Id} \end{pmatrix}^\top \begin{pmatrix} df \\ \text{Id} \end{pmatrix}} dx \quad \text{s.t. } f \in \text{Diff}^+(\Omega, \Omega)$$

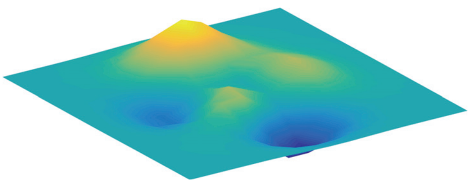
where $c(x, y) = \|\mathcal{I}_1(x) - \mathcal{I}_2(y)\|$.

Equivalently: Find optimal $f : \Omega \rightarrow \Omega$ minimizing

$$\min_{f \in \text{Diff}^+(\Omega,\Omega)} \int_{\Gamma(f)} c(\mathbf{p}) d\mathbf{p}$$

where $\Gamma(f) = \{(p, f(p)) | p \in \Omega\}$ is the graph of f.

Minimal Surface Problem



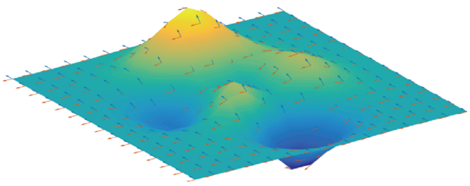
Task: The correspondence problem is equivalent to

$$\min_{\Gamma \subset \Omega\times\Omega} \int_\Gamma c(\mathbf{p}) d\mathbf{p} \quad \text{s.t. } \partial\Gamma = \emptyset \text{ inside } \Omega\times\Omega \text{ and } \pi_1(\Gamma) = \pi_2(\Gamma) = \Omega.$$

Main Challenge

How do we find a minimal 2-dimensional surface of codimension 2?

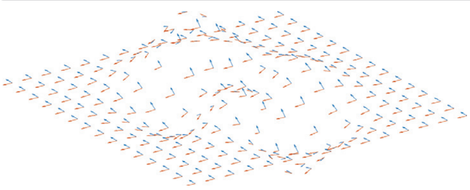
Surface Representation



Let $t : \Omega \rightarrow \Gamma$ be a parameterization of Γ:

$$\int_\Gamma c(\mathbf{p}) d\mathbf{p} = \int_\Omega c(p, t(p)) \sqrt{\det(dt(p)^\top dt(p))} dp = \int_\Omega c(p) \mathcal{A}(t_x(p), t_y(p)) dp$$

Surface Representation via 2-Vector Fields



$$\min_{\omega:\Omega^2\rightarrow\Lambda_2\mathcal{T}\mathbb{R}^4} \int_{\Omega^2} c(\mathbf{p}) \|\omega(\mathbf{p})\| d\mathcal{H}^2\mathbf{p} = \mathbf{M}(\mathcal{H}^2 \wedge c \wedge \omega)$$
$$\text{s. t. } \partial(\mathcal{H}^2 \wedge c \wedge \omega) = 0 \text{ and } \pi_1(\omega) = \pi_2(\omega) = \Omega.$$

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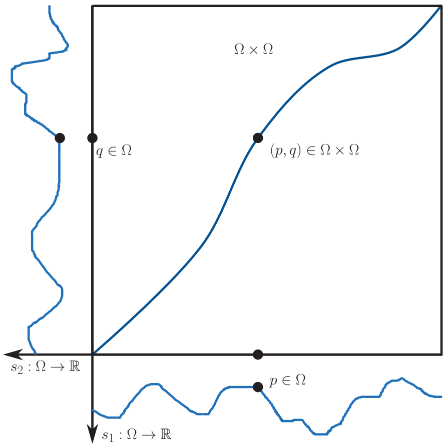
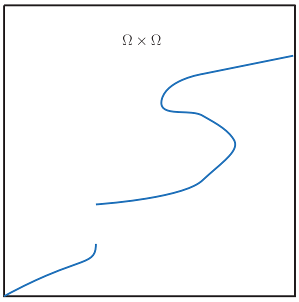
Main Contribution

We show there is a deep geometric structure underlying the spatially-regularized correspondence problem that is essential for solving the problem in a convex fashion. To the best of our knowledge we are the first that introduce this structure for image correspondence problems.

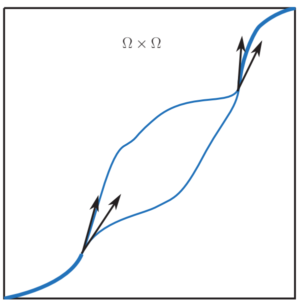
1D-Correspondence Example



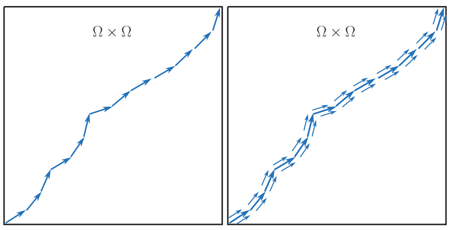
1D-Correspondence Counterexample



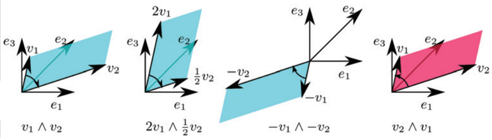
Effect of Non-Simple 2-Vectors



From Integral Currents to Smooth Normal Currents



2-Vectors

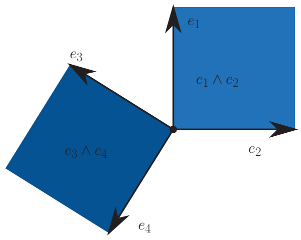


The set of 2-vectors $\Lambda_2\mathcal{T}\mathbb{R}^4$ is a vector space.

$$\dim \Lambda_2\mathcal{T}\mathbb{R}^4 = 6$$
$$\Lambda_2\mathcal{T}\mathbb{R}^4 = \text{span}\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$$

Simple 2-Vectors

- simple 2-vectors have the form $v_1 \wedge v_2 \in \Lambda_2\mathcal{T}\mathbb{R}^4$
- equivalence between simple 2-vectors and oriented 2 dimensional subspaces plus some positive area
- $\mathcal{A}(v_1, v_2) = \|v_1 \wedge v_2\|_2$



Non-Simple 2-Vectors

- $e_1 \wedge e_2 + e_3 \wedge e_4$ is non-simple
- $\|e_1 \wedge e_2\|_2 = \|e_3 \wedge e_4\|_2 = 1$, but $\|e_1 \wedge e_2 + e_3 \wedge e_4\|_2 = \sqrt{2}$.
- Solution: the *mass norm*

$$\|\omega\| = \inf\{\sum_{i=1}^N \|\omega_i\|_2 \mid \omega = \sum_{i=1}^N \omega_i, \omega_i \text{ is simple}\}$$

Convex Optimization Problem

- make coefficients of ω differentiable
- replace $\mathcal{H}^2$ by $\mathcal{H}^4$
- $\partial(\mathcal{H}^4 \wedge c \wedge \omega) = \emptyset \;\Leftrightarrow\; \text{div}\,\omega = 0$, where $\text{div}\,\omega = \sum_{i=1}^n \frac{\partial\omega}{\partial x_i} \lfloor dx_i$

Continuous Optimization Problem

$$\inf_{\omega:\Omega^2\rightarrow\Lambda_2\mathcal{T}\mathbb{R}^4} \int_{\Omega^2} c(\mathbf{p}) \|\omega(\mathbf{p})\| d\mathcal{H}^4\mathbf{p} = \mathbf{M}(\mathcal{H}^4 \wedge c \wedge \omega)$$

s. t. ω's coefficient functions are differentiable, $\text{div}\,\omega = 0$ and $\pi_1(\omega) = \pi_2(\omega) = \Omega$.

Discrete Optimization Problem

$$\min_{x\in\mathbb{R}^{6N}} \sum_i \|X_i\| \quad \text{s. t. } \text{div}\,x = \mathbf{0} \text{ and } \pi_1(x) = \pi_2(x) = \mathbf{1}.$$

Correspondence Example