3D Image Reconstruction from X-Ray Measurements with Overlap

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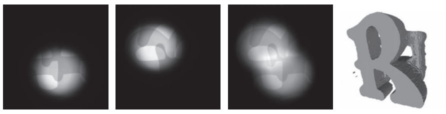
Motivation

- Goal: Reconstruct a 3D image from a set of 2D X-ray projections.
- Novel scanner devices with spatially and temporally overlapping rays yield a new type of nonlinear ray constraints.
- Applications: medical imaging, industrial inspection, airport security, ...



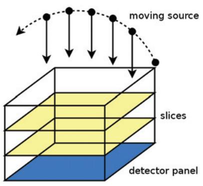
Sequential exposures: no overlapping rays | Simultaneous: with overlap | Reconstructed surface

Contributions

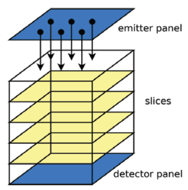
- A new image reconstruction problem from X-ray measurements with overlap.
- A proof of partial convexity.
- A new optimization method based on forward-backward splitting.
- Experimental validation with real data.

Geometry of rays

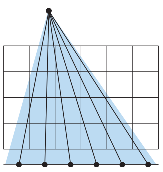
Traditional CT: a single source



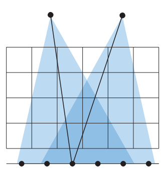
X-ray emitter array: multiple sources



Multiple (partially) simultaneously emitting sources lead to measurements with overlap:



Sequential scan: no overlap



Overlapping rays: several rays can reach the same detector at the same time

Acknowledgements

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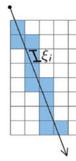
A new type of ray constraints

Sequential measurements yield linear constraints:

$$Ax = b \quad \text{with} \quad A_{ij} = -\xi_{ij}$$



Overlapping rays yield nonlinear constraints:

$$\sum_{k=1}^{p} \exp\left(\sum_{i=1}^{n} -\xi_{ijk} x_i\right) = b_j, \quad \forall j = 1, \dots, m$$

Sparse vectors of intersection lengths:

$$r_{jk} = \left(-\xi_{1jk}, \dots, -\xi_{njk}\right)^\top \in \mathbb{R}^n, \quad \forall j, k$$

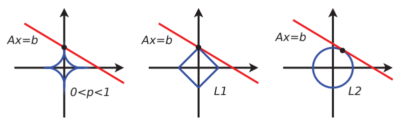
A new image reconstruction problem

Lagrangian formulation with sparsity prior:

$$\min_{x \geq 0} \underbrace{\lambda \|x\|_1}_{f(x)\text{ convex}} + \underbrace{\frac{1}{2} \sum_{j=1}^{m} \left(\sum_{k=1}^{p} \exp\left(\sum_{i=1}^{n} -\xi_{ijk} x_i \right) - b_j \right)^2}_{g(x)\text{ partially convex, differentiable}}$$

Possible choices for the regularization term:

- $\|x\|_1$: sparsity of densities
- $\|x\|_{TV}$: sparsity of gradients
- $\|\Phi x\|_1$: sparsity of wavelet frequencies



Theorem:
The optimization problem is partially convex for $x \leq \hat{x}$, and ∇g is Lipschitz continuous with Lipschitz constant $L = 2mp^2\xi_{max}^2$.

Forward-backward splitting optimization

Input:
$b \in \mathbb{R}^m$: measurements
$r_{jk} \in \mathbb{R}^n$: intersection lengths
$c \in (0,1)$: line search control parameter

Initialize $x^0 = 0$
Iterate until convergence:

1. *Compute search direction:*

$$\nabla g = \sum_{j=1}^{m} \left(\sum_{k=1}^{p} e^{r_{jk}^\top x} - b_j \right) \left(\sum_{k=1}^{p} r_{jk} e^{r_{jk}^\top x} \right)$$

2. *Backtracking line search:*

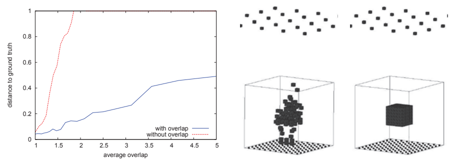
$\alpha = 1/L$
$x^{new} = \text{prox}_{\lambda f}(x^t - \alpha \nabla g)$
while $\pi(x^{new}) < b$:
  $\alpha \leftarrow \alpha c$
  $x^{new} = \text{prox}_{\lambda f}(x^t - \alpha \nabla g)$

3. *Update x:* $x^{t+1} = x^{new}$

with $\text{prox}_{\lambda f}(x) = \arg\min_y \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|_2^2 \right\}$

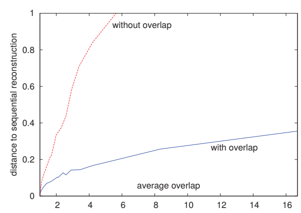
Reconstruction with ground truth data

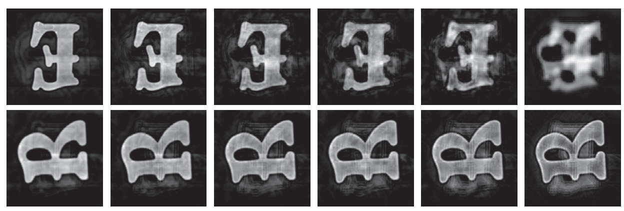
Reconstruction of a cube from simulated data and reconstructed surface for $\bar{p} \approx 2$:



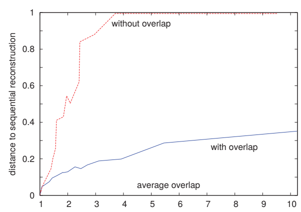
Distance to ground truth | no overlap | with overlap

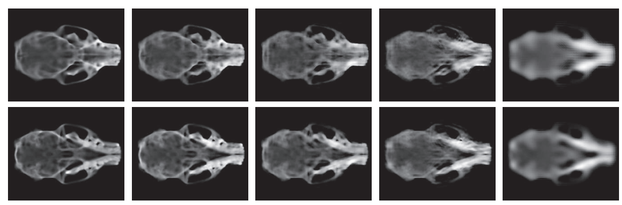
Reconstruction from real-world measurements





$\bar{p} = 1$ | $\bar{p} = 2.06$ | $\bar{p} = 4.19$ | $\bar{p} = 5.58$ | $\bar{p} = 8.34$ | $\bar{p} = 16.67$





$\bar{p} = 1$ | $\bar{p} = 1.85$ | $\bar{p} = 3.92$ | $\bar{p} = 5.45$ | $\bar{p} = 10.23$

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