Uncovering Symmetries in Polynomial Systems

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Overview

In this paper we study symmetries in polynomial equation systems and how they can be integrated into the action matrix method.

- We generalize the partial p-fold symmetry from [1, 2]
- We show how to use multiple independent symmetries.
- We provide a simple method for finding *hidden* symmetries.
- We show two examples where symmetry allows for more compact solvers.

Prior Work

Consider

$$\begin{cases} x^2 + y - 2 = 0, \\ x^2y^2 - 1 = 0 \end{cases} \quad \mathcal{V} = \{(\pm 1, 1),\ (\pm\varphi, -\varphi^{-1}),\ (\pm\varphi^{-1}, \varphi)\}. \tag{1}$$

- Two-fold sign symmetry in x-variable, i.e. $(x, y) \in \mathcal{V} \implies (-x, y) \in \mathcal{V}$,
- This type of symmetry was studied in [1, 2].

Definition 1. *The polynomial $f(\mathbf{x}, \mathbf{y})$ has a **partial p-fold symmetry** in $\mathbf{x}$ if the sum of the exponents for $\mathbf{x}$ of each monomial has the same remainder q modulo p, i.e.*

$$f(\mathbf{x}, \mathbf{y}) = \sum_k a_k \mathbf{x}^{\alpha_k} \mathbf{y}^{\beta_k} \implies q \equiv \mathbb{1}^T \alpha_k \bmod p \quad \forall k. \tag{2}$$

Weighted Symmetries

Consider

$$\begin{cases} x^3 - 1 = 0, \\ xy - 1 = 0 \end{cases} \quad \mathcal{V} = \{(1, 1),\ (\frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}),\ (\frac{1 - i\sqrt{3}}{2}, \frac{1 + i\sqrt{3}}{2})\}. \tag{3}$$

- No partial p-fold symmetry, but $(x, y) \in \mathcal{V} \implies (e^{2\pi i \frac{1}{3}} x, e^{2\pi i \frac{2}{3}} y) \in \mathcal{V}$.

Definition 2. *The polynomial $f(\mathbf{x})$ has a **weighted p-fold symmetry** with weights $\mathbf{c} \in \mathbb{Z}_p^n$ if the $\mathbf{c}$-weighted sum of the exponents for $\mathbf{x}$ of each monomial has the same remainder q modulo p, i.e.*

$$f(\mathbf{x}) = \sum_k a_k \mathbf{x}^{\alpha_k} \implies q \equiv \mathbf{c}^T \alpha_k \bmod p \quad \forall k. \tag{4}$$

- Weighted sum of exponents should have same remainder.
- For the example in (3) we have $\mathbf{c} = (1, 2)$, $p = 3$.
- Then the solution set $\mathcal{V}$ satisfies

$$(x_1, x_2, \ldots, x_n) \in \mathcal{V} \implies (e^{2\pi i \frac{c_1}{p}} x_1, e^{2\pi i \frac{c_2}{p}} x_2, \ldots, e^{2\pi i \frac{c_n}{p}} x_n) \in \mathcal{V}.$$

- Partial p-fold symmetry corresponds to binary weights.

Block Diagonal Action Matrices

Action matrix will be **block diagonal** if the action polynomial is invariant to the symmetry.

$$\begin{cases} x^2 + y^2 - 2 = 0, \\ xy^2 - x = 0. \end{cases} \quad \begin{matrix} (x, y) \to (-x, y), & \mathbf{c}_1 = (1, 0), & p = 2, \\ (x, y) \to (x, -y), & \mathbf{c}_2 = (0, 1), & p = 2. \end{matrix}$$

Basis for quotient space given by

$$\mathcal{B} = \{1,\ x,\ y,\ xy,\ y^2,\ y^3\} \tag{5}$$

Grouping basis elements by weighted exponents w.r.t. $\mathbf{c}_1$ and $\mathbf{c}_2$ modulo 2

$$\mathcal{B}_{0,0} = \{1,\ y^2\},\ \mathcal{B}_{0,1} = \{y,\ y^3\},\ \mathcal{B}_{1,0} = \{x\},\ \mathcal{B}_{1,1} = \{xy\}.$$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & 2 & & \\ & & -1 & 2 & & \\ & & & & -1 & 2 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} xy \\ x \\ y^3 \\ y \\ y^2 \\ 1 \end{bmatrix} = x^2 \begin{bmatrix} xy \\ x \\ y^3 \\ y \\ y^2 \\ 1 \end{bmatrix}, \tag{6}$$

Idea is to compute solutions from only one of the blocks. $\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y^2 \\ 1 \end{bmatrix} = x^2 \begin{bmatrix} y^2 \\ 1 \end{bmatrix}$.

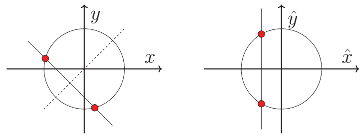
Hidden Weighted Symmetries

For $\alpha \in (-\sqrt{2}, \sqrt{2})$ consider the following family of polynomial systems

$$\begin{cases} x^2 + y^2 = 1, \\ x + y = \alpha \end{cases} \quad (x, y) \to (y, x) \tag{7}$$

No weighted symmetries. Change of variables reveals a 2-fold symmetry in one of the variables.

$$\begin{cases} \hat{x} = x + y \\ \hat{y} = x - y \end{cases} \implies \begin{cases} \frac{1}{2}\hat{x}^2 + \frac{1}{2}\hat{y}^2 = 1 \\ \hat{x} = \alpha \end{cases}$$



This can be done in general if the solutions are stable under some linear transform. In the paper we present a simple method for finding these symmetries and the correct change of variables.

Weak Perspective-n-Points

Pose estimation in scaled orthographic camera

$$\min_{s,R} \ \|RA - B\|_F^2 \quad \text{s.t.} \quad RR^T = s^2 I_2, \quad R \in \mathbb{R}^{2\times 3}. \tag{8}$$

Quaternion representation for scaled rotation

$$R(q) = \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2q_3 - q_1q_4) & 2(q_1q_3 + q_2q_4) \\ 2(q_1q_4 + q_2q_3) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3q_4 - q_1q_2) \end{bmatrix}. \tag{9}$$

Finding minimum by looking at stationary points

$$\nabla_{\mathbf{q}} f(\mathbf{q}) = 0, \quad \text{where} \quad f(\mathbf{q}) = \|R(\mathbf{q})A - B\|^2. \tag{10}$$

Rotation parameterization is invariant under sign change and

$$R(q_1, q_2, q_3, q_4) = R(iq_4, iq_3, -iq_2, -iq_1). \tag{11}$$

Perform change of variables

$$\hat{\mathbf{q}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & -i & 0 \\ -1 & 0 & 0 & 1 \\ i & 0 & 0 & i \\ 0 & -1 & 1 & 0 \end{bmatrix} \mathbf{q}. \tag{12}$$

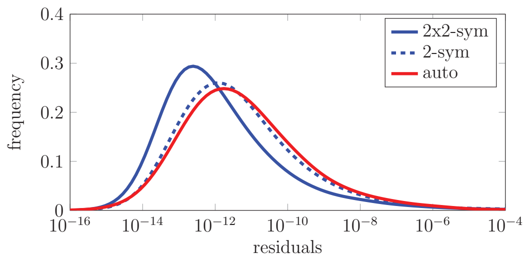
In these new variables the rotation matrix becomes

$$R(\hat{\mathbf{q}}) = \begin{bmatrix} \hat{q}_1^2 - \hat{q}_2^2 - \hat{q}_3^2 + \hat{q}_4^2 & -i\hat{q}_1^2 - i\hat{q}_2^2 + i\hat{q}_3^2 + i\hat{q}_4^2 & 2(\hat{q}_1\hat{q}_2 + 2\hat{q}_3\hat{q}_4) \\ -i\hat{q}_1^2 + i\hat{q}_2^2 - i\hat{q}_3^2 + i\hat{q}_4^2 & -\hat{q}_1^2 - \hat{q}_2^2 - \hat{q}_3^2 - \hat{q}_4^2 & 2i(\hat{q}_3\hat{q}_4 - \hat{q}_1\hat{q}_2) \end{bmatrix}. \tag{13}$$

Symmetries in original parameterization transformed into two 2-fold partial symmetries

$$\mathbf{c}_1 = (1, 1, 0, 0) \quad \mathbf{c}_2 = (0, 0, 1, 1). \tag{14}$$

	2x2-sym	2-sym	Kukelova et al. [3]
Elimination template	104×90	234×276	243×276
Action matrix	8×8	16×16	33×33



References

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[2] Kuang, Y., Zheng, Y., Åström, K.: Partial symmetry in polynomial systems and its applications in computer vision. In: The IEEE Conference on Computer Vision and Pattern Recognition (CVPR). (June 2014)

[3] Kukelova, Z., Bujnak, M., Pajdla, T.: Automatic generator of minimal problem solvers. In: Proceedings of the European Conference on Computer Vision (ECCV). Springer (2008) 302–315